Math 206A Lecture 25 Notes

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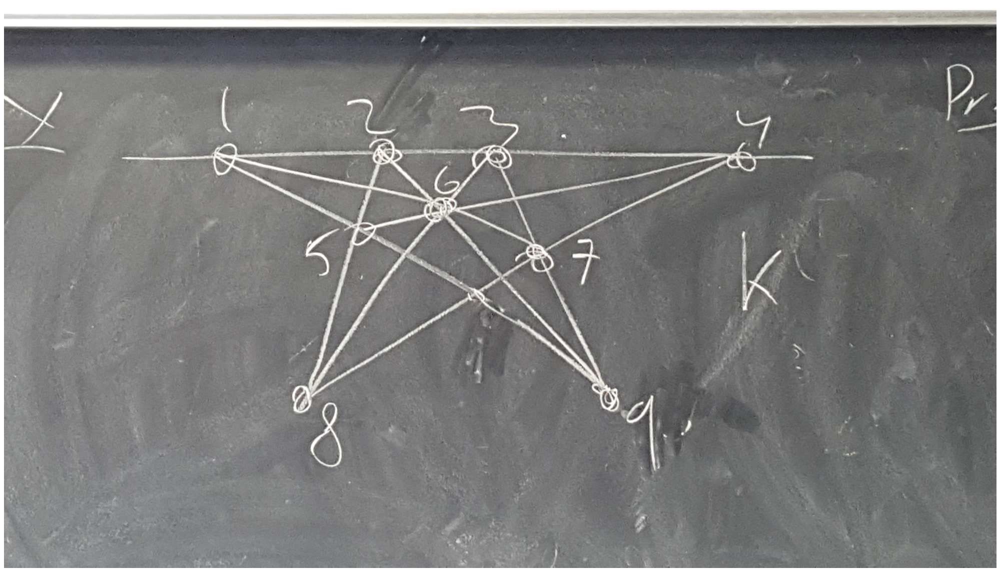
1 Irrational Point and Line Configurations, and Lawrence's Construction

1.1 Irrational point and line confingurations

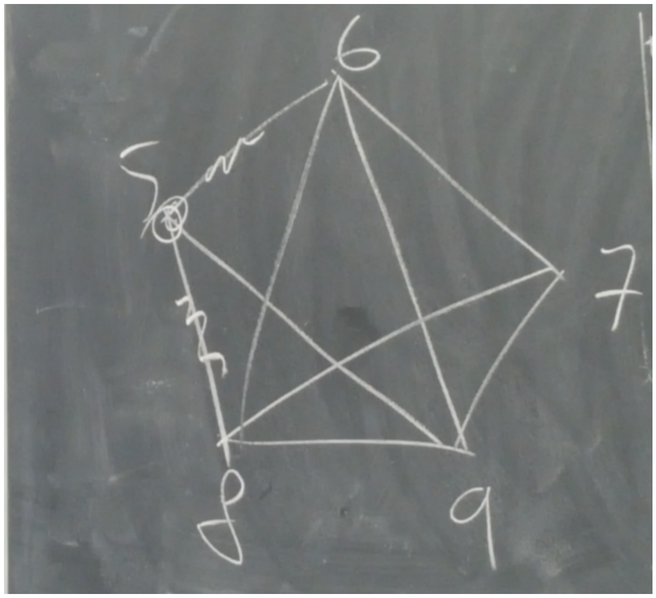
Theorem 1.1 (Perles, 1970s)**.** *There exists $d > 3$ and a convex polytope $P \subseteq \mathbb{R}^d$ such that for all $P \subseteq \mathbb{Q}^d$, $\alpha(P) \not\simeq \alpha(P')$.*

Theorem 1.2. *There exists a point and line configuration $K = (V, L)$ realizable over $\mathbb{R}$ but not over $\mathbb{Q}$.*

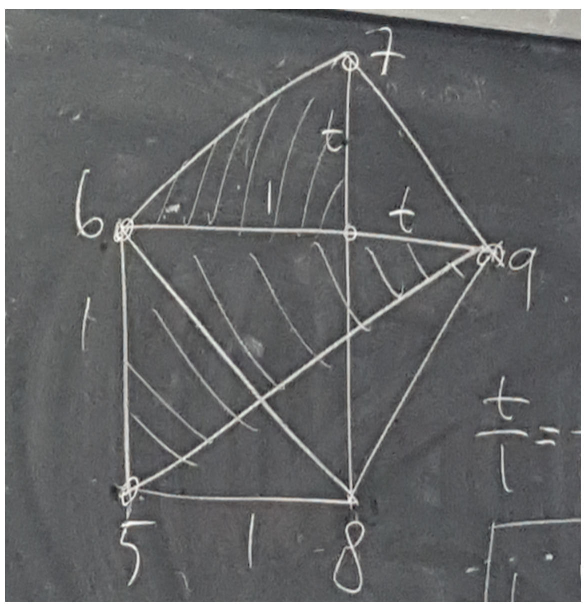
Example 1.1. Here is an actual example. Start with the configuration



Send the line passing through $1, 2, 3, 4$ to ∞. We get the following picture,



where 56 || 78, 69 || 58, 59 || 67, and 58 || 69. Specifically, we set the edge lengths as follows:



From similar triangles, we get $t/1 = 1/(1+t)$. So $t = (\sqrt{5}-1)/2$.

1.2 Lawrence's construction

Theorem 1.3 (Lawrence). *Let $K = (V, L)$ be irrational with $|V| = n$. Then there exists a convex polytope $P \subseteq \mathbb{R}^d$ with $d = n + 3$ and $2n$ vertices such that P is irrational.*

Proof. Let $f : V \to \mathbb{R}^2$ be a realization of K. Let $w = (f(v_i), 1) \in \mathbb{R}^3$. Consider the $2n$ points $x_i = (w_i, e_i), y_i = (w_i, we_i)\mathbb{R}^{3+n}$, where $\mathbb{R}^n = \langle e_1, \ldots, e_n \rangle$. Now let $P = \operatorname{conv}(\{x_i, y_i : i = 1, \ldots, n\})$. Then P has all x_i, y_i as vertices. P is irrational.

To show that P is irrational, suppose that $\sum_{i=1}^n \alpha_i x_i + \beta_i y_i = 0$. Then $\alpha_i = -2\beta_i$ for all i. Then $\sum_{i=1}^n (-2\beta_i) w_i + (\beta_i) w_i = 0$, so $\sum_{i=1}^n \beta_i w_i = 0$. If some of the w_i lie on a line, then we get such a linear relation. Then P has some 5-dimensional faces (containing $x_i, y_i, x_j, y_j, x_k, y_k$), not just 6-dimensional faces. Then v_i, v_j, v_k lie on the same line. $\square$

1.3 Constructing regular n-gons

Gauss was interested in the following question: For which n does there exist a ruler and compass construction of the regular n-gon?

Theorem 1.4 (Gauss). *If $n = 7$, there is no ruler and compass construction of the regular n-gon.*

Theorem 1.5 (Gauss). *If $n = 17$, there is a ruler and compass construction of the regular n-gon.*